

THE SCHREIER CONTINUUM AND ENDS

ALEX CLARK, ROBERT FOKKINK, AND OLGA LUKINA

ABSTRACT. Blanc showed in his thesis that a compact minimal foliated space with a residual subset of 2-ended leaves can contain only 1 or 2 ended leaves. In this paper we give examples of compact minimal foliated spaces where a topologically generic leaf has 1 end, there is an uncountable set of leaves with 2 ends and a leaf with $2n$ ends, for a given $n > 1$. The examples we present are weak solenoids, which allows us to represent the graph of the group action on the fibre as the inverse limit of finite coverings of a finite graph, which we call the Schreier continuum, which we use to obtain the result. While in certain cases the problem can be reduced to the study of a self-similar action of an automorphism group of a regular tree, our geometric technique is more general, as it applies to cases where the action is not self-similar.

1. INTRODUCTION

A compact foliated space X is a compact metrisable space together with a decomposition of X into subsets S_α , called *leaves*, such that for each $x \in X$ there is a chart $\varphi_x : U_x \rightarrow \mathbb{R}^n \times Z$, where Z is a compact metrisable space, and the restriction of φ_x onto a connected component of $S_\alpha \cap U$ is constant on the second component [7]. One class of examples of foliated spaces arising in continuum theory are weak solenoids; i.e., the inverse limits $X_\infty = \lim_{\leftarrow} \{f_{i-1}^i : X_i \rightarrow X_{i-1}\}$, where X_i is a closed manifold, and each bonding map f_{i-1}^i is a finite-to-one covering map [14].

Weak solenoids are natural generalisations of the well-known one-dimensional solenoids and have the structure of a foliated bundle over a closed manifold with a Cantor set fibre. Leaves in solenoids are their path-connected components, and every leaf in a solenoid is dense. One-dimensional solenoids are important in the study of flows and have been identified as minimal sets of smooth flows [5, 19, 18, 22, 33] and are known to occur generically in Hamiltonian flows on closed manifolds [23]. The more general solenoids introduced in [25] provide a class of higher-dimensional continua that have been the object of great interest in the study of continua [28, 30, 14]. It was shown in [12] that the equicontinuous minimal sets of foliations having the local structure of a disk in a leaf times a Cantor set are solenoids, and in [11] examples are provided of certain classes of solenoids embedded as minimal sets of smooth foliations of manifolds. End structures of leaves provide the means to study their behaviour at infinity, and so are important for the study of the inner structure of solenoids.

Ghys shows in his seminal work [17] that the leaves of almost all (with respect to an appropriate harmonic measure) points of a compact foliated space have either 0,

2000 *Mathematics Subject Classification.* Primary 57R30, 37C55, 37B45; Secondary 53C12.

Authors supported by NWO travel grant 040.11.132.

AC and OL supported in part by EPSRC grant EP/G006377/1.

1, 2 or a Cantor set of ends. The harmonic measures used are those introduced by Garnett [16] and locally have the form of the product of a harmonic function and the Riemannian volume on the plaques of the leaves in foliation charts. These measures are used in foliation theory since, in contrast to invariant transverse measures, they always exist and at the same time many of the basic results from ergodic theory carry over to these harmonic measures. Inspired by this result, Cantwell and Conlon [9] obtained similar results in the topological setting. They show that for a complete foliated space with a totally recurrent leaf (in particular, for compact minimal sets), the leaves of all points of a residual subset either have no ends, one end, two ends or a Cantor set of ends. Blanc [1] showed that if a compact minimal foliated space X has a topologically generic leaf with 2 ends, then any other leaf in X has either 1 or 2 ends. In this paper we show that if a compact foliated space has a topologically generic leaf with 1 end, then the maximal number of ends can be made an arbitrarily large finite number. In particular, we prove the following result.

Theorem 1.1. *Given $n > 1$, there exists a weak solenoid X_n where the union of leaves with 1 end forms a residual subset, the union of leaves with 2 ends forms a meager subset and the number of such leaves is uncountable, and there is a single leaf with $2n$ ends.*

For each $n > 1$ in Theorem 1.1 an example of a foliated space with required properties is obtained by a suitable modification of the Schori solenoid [30]. The Schori solenoid is the inverse limit of an inverse sequence of non-regular 3-to-1 coverings of a genus 2 surface X_0 and, as any solenoid, is a foliated fibre bundle with the base X_0 . An important property is that every leaf L in a foliated bundle $p : E \rightarrow X_0$ is a covering space of the base, and so by the result of Scott [31, Lemma 1.2] the number of ends of L is the same as the number of ends of the pair of groups $(\pi_1(X_0, p(x)), p_*\pi_1(L, x))$, or the number of ends of the Schreier diagram of the coset space $\pi_1(E, p(x))/p_*\pi_1(L, x)$. The Schreier space of E is then the set Λ_∞ of all Schreier diagrams of leaves, bound together in a suitable topology so that they preserve the essential dynamical properties of E . In the case when $E = X_\infty$ is a weak solenoid, there are intermediate coverings $E \xrightarrow{p_i} X_i \xrightarrow{p_0^i} X_0$ which allow us to obtain a representation of Λ_∞ as the inverse limit of finite graphs, and make computations by studying the projections of path components of Λ_∞ onto these finite graphs.

It is interesting to compare these results with the recent results [3, 13], where the number of ends of a large class of orbital Schreier graphs arising in the context of self-similar group actions is determined. There, general results are obtained, showing that in a measure-theoretic sense most orbital Schreier graphs have either one or two ends, and for most self-similar groups the generic end structure is one. In the case when a weak solenoid X_∞ has a representation $X_\infty = \varprojlim \{p_{i-1}^i : X_i \rightarrow X_{i-1}\}$ such that every covering map p_{i-1}^i has the same degree and does not significantly differ from the other covering maps in the sequence, the image of the monodromy representation $\pi_1(X_0, p(x)) \rightarrow \text{Homeo}(F)$, where $F \cong p^{-1}(p(x))$ is a fibre of the weak solenoid, can be identified with a group of automorphisms of a regular tree. Such an action is self-similar, and is generated by a finite automaton. However, the class of weak solenoids is a lot richer than those with self-similar action of the monodromy group, and our geometric approach is applicable to any of them. In Section 3.6 we compute end structures for a weak solenoid which is the inverse

limit of a sequence of covering maps of variable degree, where a map of each degree occurs an infinite number of times. As is well-known, for solenoids of dimension 1, i.e. inverse limits of finite coverings of \mathbb{S}^1 , equivalence classes of sequences of degrees provide a classification up to a homeomorphism, and for dimension higher than 1 determining whether a weak solenoid has a representation as the inverse limit of coverings of constant degree is by no means a trivial question. Our geometric approach overcomes this difficulty.

The results of [3, 13] reveal connections of our work to the study of Julia sets, and leads to interesting questions concerning the interplay between topological and measure-theoretical genericity of various properties of foliated manifolds, such as those in Section 4.

In detail, our paper is organised as follows. In Section 2 we recall some background knowledge from algebraic topology and foliation theory, and give a brief outline of the computation technique of the Schreier continuum. Section 3.2 contains computations of the end structures for some specific solenoids. Section 4 indicates some open questions.

2. PRELIMINARIES: ENDS OF COVERING SPACES AND FOLIATED BUNDLES

The set of ends of a non-compact separable metric topological space X is a set of ideal points at infinity that compactify X . Ends appear first in the work of Freudenthal [15], and the compactification by ends is often called the *Freudenthal compactification*. More precisely [7], given a sequence of compact subsets

$$K_1 \subset K_2 \subset K_3 \subset \cdots \quad \text{with} \quad \bigcup_i K_i = X$$

consider chains of unbounded connected components of $X \setminus K_i$

$$\{U_i\} : U_1 \supset U_2 \supset U_3 \supset \cdots \quad \text{where} \quad \bigcap_i U_i = \emptyset.$$

Set $\{U_i\} \sim \{V_j\}$ if for any i there is a $j > i$ such that

$$(U_i \cap V_i) \supset (U_j \cup V_j).$$

Then an *end* e of X is an equivalence class of chains $\{U_i\}$. Denote by $\mathcal{E}(X)$ the set of ends, and set $X^* = X \cup \mathcal{E}(X)$. Put a topology on X^* by saying that sets open in X are open in X^* , and if $\{U_i\}$ is a sequence representing an end e , then U_i together with all ends contained in U_i is open. With this topology, X^* is compact, X is open in X^* and $\mathcal{E}(X)$ is a totally disconnected subset of X^* [7].

2.1. Ends of covering spaces. Recall (see, e.g. [4]) that given a group G generated by $S = \{s_1, \dots, s_n\}$, the *Cayley graph* Γ is the graph whose vertices are the elements of G and for which there is an edge labelled by $s_i \in S$ joining g to g' if and only if $g' = g s_i$. Then G acts on the left of Γ in a natural way. Given a subgroup C of G the *Schreier diagram* of (G, C) is the orbit space Γ/C of the left action of the subgroup C on Γ . While the actual construction of this graph depends on the choice of generators S , the number of ends of the graph is independent of this choice (see, e.g. [4]).

In his classic work Hopf [21] showed that a non-compact regular covering space of a compact polyhedron has either one (e.g. the plane), two (e.g. the line) or a Cantor set of ends (e.g. the universal cover of the figure eight). Here a covering space $p : L \rightarrow B$ is regular if $p_*\pi_1(L, x)$ is a normal subgroup of $\pi_1(B, p(x))$, and

geometrically the number of ends of L coincides with that of the Cayley diagram of the quotient group $\pi_1(B, p(x))/p_*\pi_1(L, x)$. In the case when $p : L \rightarrow B$ is non-regular, the generalisation of Hopf's theorem to irregular coverings (see, e.g., Scott [31, Lemma 1.1 - 1.2]) yields that the number of ends of L is the same as the number of ends of the Schreier diagram of the coset space $\pi_1(B, p(x))/p_*\pi_1(L, x)$.

2.2. Ends of leaves in foliated bundles. Let E be a compact metrisable space with foliation \mathcal{F} , that is [7], there is a decomposition of E into disjoint subsets $\{L_\alpha\}_{\alpha \in A}$ called leaves of the foliation \mathcal{F} , such that each $x \in E$ has an open neighborhood U_x with a homeomorphism $\varphi_x : U_x \rightarrow V_x \times Z_x$, where $V_x \subset \mathbb{R}^n$ and Z_x is a compact metrisable space, and connected components of $L_\alpha \cap U_x$ are given by fixing values in Z_x . The sets $\varphi_x^{-1}(V_x \times \{z\})$, $z \in Z_x$, are open sets of the *leaf topology* on L_α . If the transition maps $\varphi_x \circ \varphi_y^{-1}$ are differentiable with respect to coordinates on \mathbb{R}^n , then the leaves L_α are smooth manifolds, and \mathcal{F} is a smooth foliation.

Suppose there is a locally trivial projection $p : E \rightarrow B$ on a closed manifold B , such that leaves of \mathcal{F} are transverse to the fibres of p . Then $p : E \rightarrow B$ is a *foliated bundle* (see, e.g., Candel and Conlon [7, Example 2.1.5]), and, in particular, for each leaf $L \subset E$ the restriction $p|_L : L \rightarrow B$ in the leaf topology is a covering space. Given $b \in B$, there is a *total holonomy* homomorphism

$$h : \pi_1(B, b) \rightarrow \text{Homeo}(F)$$

determined by the lifting of loops in B based at b to paths contained in the leaves of \mathcal{F} . This allows one to translate many questions about \mathcal{F} into questions involving subgroups of $\text{Homeo}(F)$. Associated to each point $x \in F$ is the subgroup of $\pi_1(B, b)$ corresponding to the stabiliser of x for the induced action on F , which we will refer to as the *kernel* of x and denote \mathcal{K}_x .

In particular, if L_x is a leaf containing x , then by standard algebraic topological arguments (see, e.g., [24]) we have $\mathcal{K}_x = p_*\pi_1(L_x, x)$, and the leaf L_x is homeomorphic to the quotient \tilde{B}/\mathcal{K}_x , where \tilde{B} is the universal cover of B . It is clear that if $y \in L_x$, then $\mathcal{K}_x = \mathcal{K}_y$.

When the foliated bundle $p : E \rightarrow B$ is principal (that is, when the bundle automorphisms act transitively on the fibres of p), then all the leaves of \mathcal{F} are regular coverings of B and so Hopf's theorem applies. Moreover, in this case the bundle automorphisms yield a homeomorphism between any two leaves and so all leaves have homeomorphic end spaces, either 0 (compact), 1, 2 or a Cantor set. If $p : E \rightarrow B$ is not principal, the kernel \mathcal{K}_x , $x \in E$, may vary depending on the point.

2.3. The Schreier space of a foliated bundle. We construct the *Schreier space* of a foliated bundle $p : E \rightarrow B$, where B is a closed manifold. We fix a finite set of generators $S = \{[\gamma_1], \dots, [\gamma_n]\}$ of $\pi_1(B, b)$ that are represented by loops γ_i in B that intersect pairwise only at the base point b . If the dimension of B is 2 we can choose the generators to be given in the standard way by the boundary edges of a polygon whose quotient is homeomorphic to B , and when the dimension of B is greater than 2 it is not hard to see how to construct these loops given any finite set of generators and local charts.

Definition 2.1. *For a given foliated bundle $p : E \rightarrow B$ with a closed manifold B as base and generators $S = \{[\gamma_1], \dots, [\gamma_n]\}$ of $\pi_1(B, b)$ and fibre F as above, the*

Schreier space \mathcal{S} is the subspace of E formed by taking the union of all lifts of the paths $\{\gamma_1, \dots, \gamma_n\}$ to paths in E starting at points of F .

It is then straightforward that $L_x \cap \mathcal{S}$ in the leaf topology is the Schreier graph of the coset space $\pi_1(B, b)/\mathcal{K}_x$, and so has the same number of ends as the leaf L_x .

We shall refer to the sets of the form $L \cap \mathcal{S}$ for a leaf L of \mathcal{F} as *leaves* of \mathcal{S} and a set formed by a union of leaves will be called *saturated* as with foliations. A compact saturated subset of \mathcal{S} is *minimal* if it is the closure of each of its leaves. If the sequence $\{x_i\}_{i=1}^\infty$ converges to x in \mathcal{S} , then the corresponding plaques in a foliation chart of x converge to the plaque of x . Hence, since the leaves of \mathcal{S} are formed by lifting paths in B , a set is saturated in \mathcal{S} if and only if the corresponding set is saturated in E and a set in \mathcal{S} is minimal if and only if the corresponding set is a compact minimal subset of E . Since the leaves of \mathcal{S} are connected, minimal sets are compact and connected; that is, they are continua. Thus, we refer to a minimal subcontinuum Λ of \mathcal{S} as the *Schreier continuum* of the corresponding minimal set of E . If the foliation of the bundle $p : E \rightarrow B$ is minimal, the notions of the Schreier space and the Schreier continuum are interchangeable.

3. SCHREIER CONTINUA FOR SOLENOIDS

In this section we prove Theorem 1.1, that is, for any $n > 1$ we construct a compact foliated space X_n such that the union of leaves with 1 end is a residual subset of X_n , the union of leaves with 2 ends forms a meager subset of X_n and the number of such leaves is uncountable, and there is a single leaf with $2n$ ends.

We are going to construct these examples as inverse limits of sequences of finite coverings of a genus 2 surface, i.e. *weak solenoids*. For convenience we recall basic facts about solenoids in Section 3.1, where we also give some algebraic examples of computations of end structures. The proof of Theorem 1.1 is obtained on the basis of the Schori solenoid [30], an example where a direct algebraic technique is not sufficient.

The proof of Theorem 1.1 for $n = 2$ is detailed in Section 3.2. In Section 3.4 we show how to modify Schori's solenoid in such a way that the exceptional leaf has any given even number of ends. In Section 3.5 we describe the Schreier continuum for the Rogers-Tollefson example (see also Example 3.2). In Section 3.6 we give an example of a solenoid, based on the Schori construction, where the group of automorphisms of the fibre acts non self-similarly.

3.1. Weak solenoids as foliated bundles. A general class of examples of foliated bundles with a Cantor set fibre can be found in weak solenoids. Here, a solenoid X_∞ will refer to the inverse limit

$$(3.1) \quad X_\infty = \varprojlim \{X_k, f_{k-1}^k, \mathbb{N}_0\}$$

of an inverse sequence of closed manifolds X_k where each bonding map $f_{k-1}^k : X_k \rightarrow X_{k-1}$ is a covering map of index greater than one. The projection $f_0 : X_\infty \rightarrow X_0$ is a fibre bundle projection with a profinite structure group and Cantor set fibre (see Fokkink and Oversteegen [14, Theorem 33]). When a solenoid is homogeneous (i.e., when the homeomorphism group acts transitively on the solenoid), the solenoid can be represented as the inverse limit of an inverse sequence where each covering map f_0^k is regular [14], and in this case the associated bundle is principal (see McCord [25, Theorem 5.6]). Thus, a solenoid can be considered a natural generalisation of

covering spaces of closed manifolds where the fibre is no longer discrete but instead totally disconnected, and where regular covering spaces correspond to homogeneous solenoids. Locally, a solenoid is homeomorphic to $D \times \text{Cantor Set}$, where D is a Euclidean disk of the same dimension as the base manifold X_0 .

Denote by G_k the image of the homomorphism

$$(f_0^k)_* : \pi_1(X_k, x_k) \rightarrow \pi_1(X_0, x_0),$$

then the kernel at $x_\infty = (x_k)$, as defined in the previous section, is given by

$$\mathcal{K}_{x_\infty} = \bigcap_{k \in \mathbb{N}_0} G_k.$$

Thus, the Schreier continuum associated to X_∞ and some fixed finite set of generators S of $\pi_1(X_0, x_0)$ is given by

$$\Lambda = \varprojlim \{\Lambda_k, \nu_{k-1}^k, k \in \mathbb{N}_0\}.$$

where Λ_k is the Schreier diagram for the coset space $\pi_1(X_0, x_0)/G_k$. Alternatively, consider Λ_0 to be the image of the Schreier space \mathcal{S} under the covering projection f_0 , and each Λ_k as the lift of Λ_0 to X_k by the covering map f_0^k .

Example 3.1. In the case that the solenoid is a principal bundle and all path-connected components are homeomorphic, the end structure can typically be found directly by analysing the kernels \mathcal{K}_x at points $x \in F$, and all possible end structures do occur. For example, let $X_k = \mathbb{S}^1$ and each f_{k-1}^k be a 2-fold covering map. The inverse limit

$$\Sigma_2 = \varprojlim \{\mathbb{S}^1, f_{k-1}^k, \mathbb{N}_0\}$$

is usually called the dyadic solenoid. The universal cover of each leaf is \mathbb{R} and for each $x \in \Sigma_2$ the kernel \mathcal{K}_x is trivial. Therefore, each leaf in Σ_2 is homeomorphic to \mathbb{R} and so has 2 ends. The dyadic solenoid has a self-similarity structure and its ends could be treated in the context of self-similar group actions as in [3]; however, if the bonding maps f_{k-1}^k are distinct primes and the resulting solenoid has no self-similarity, the results of [3] do not apply but this solenoid can be treated by our technique in the same way as the dyadic solenoid.

Generalising this example to 2 dimensions one obtains a solenoid $\Sigma_2 \times \Sigma_2$ which fibres over a 2-torus $\mathbb{S}^1 \times \mathbb{S}^1$. The universal cover of each leaf is \mathbb{R}^2 , and for each $x \in \Sigma_2 \times \Sigma_2$ the kernel \mathcal{K}_x is trivial and so each leaf has 1 end. It is known that a free group on 2 generators F_2 can be realised as the fundamental group of a 3-manifold, and F_2 is geometrically residually finite (see Scott [32] for general results about residually finite groups). Thus, choosing an appropriate decreasing chain of subgroups G_i of finite index with $\bigcap_i G_i = \emptyset$ with each G_i normal in F_2 , one can construct a solenoid in which each leaf has a Cantor set of ends.

Example 3.2. More interesting examples of ends occur when the solenoid does not have the structure of the principal bundle and the structure of the set of ends vary from leaf to leaf. One such example is given by Rogers and Tollefson [28]. Their solenoid is the inverse limit of 2-fold coverings of the Klein bottle K_i by itself

$$K_\infty = \varprojlim \{K_i, f_{i-1}^i, \mathbb{N}_0\}.$$

The solenoid K_∞ is double covered by the solenoid $\mathbb{S}^1 \times \Sigma_2$ with non-trivial covering transformation of the covering map

$$p_\infty : \mathbb{S}^1 \times \Sigma_2 \rightarrow K_\infty$$

represented by the involution $\beta \times \alpha_\infty : \mathbb{S}^1 \times \Sigma_2 \rightarrow \mathbb{S}^1 \times \Sigma_2$, where β is the rotation in the circle by π and α_∞ is the map induced by the reflections $\alpha_i : \mathbb{S}^1 \rightarrow \mathbb{S}^1$. The involution $\beta \times \alpha_\infty$ has a unique fixed point x , so it fixes a single path-connected component of $\mathbb{S}^1 \times \Sigma_2$ and is a homeomorphism when restricted to any other path-connected component. Thus every path-component in K_∞ other than the exceptional one is homeomorphic to $\mathbb{S}^1 \times \mathbb{R}$ and has 2 ends. The exceptional path-component is homeomorphic to the quotient space

$$\mathbb{S}^1 \times \mathbb{R} / \{(t, s) \sim (\beta(t), -s)\},$$

it is non-orientable and has 1 end.

3.2. Schori solenoid. The Schori example is the solenoid

$$X_{\text{Sch}} = \varprojlim \{X_k, f_k^{k+1}, \mathbb{N}_0\},$$

where X_0 is a genus 2 surface, and for every $k \in \mathbb{N}_0$ the bonding map $f_k^{k+1} : X_{k+1} \rightarrow X_k$ is a 3-to-1 non-regular covering projection. We recall briefly the construction from [30].

The construction is by induction. Let X_0 be a genus 2 surface. For $n > 0$ let an n -handle be a 2-torus with n mutually disjoint open disks taken out, then X_0 is a union of two 1-handles H_0 and F_0 which intersect along their boundaries. Let C_0 and D_0 be simple closed curves in H_0 and F_0 respectively (see Figure 1, a)). For $k > 0$ let X_k be a genus $m_k = 3^k + 1$ surface with two 2^k -handles H_k and F_k distinguished, and let two simple closed curves C_k and D_k be chosen in H_k and F_k respectively. Then the next component in the sequence is obtained in the following manner: take out the curves C_k and D_k from X_k and pull the cut handles apart by an appropriate homeomorphism to obtain a cut surface \hat{X}_k . Let $\bar{X}_k = \text{Cl}(\hat{X}_k)$, and denote by C_k' and C_k'' (resp. D_k' and D_k'') the boundary circles in H_k (resp. F_k) (see Figure 1, b)). Consider three copies \bar{X}_k^1 , \bar{X}_k^2 and \bar{X}_k^3 of \bar{X}_k , so that for $i = 1, 2, 3$ \bar{H}_k^i and \bar{F}_k^i are the cut handles in \bar{X}_k^i , and $C_k^{i'}$ and $C_k^{i''}$ (resp. $D_k^{i'}$ and $D_k^{i''}$) are the boundary circles in H_k^i (resp. F_k^i). Then in H_k^i -handles identify $C_k^{1'}$ with $C_k^{1''}$, $C_k^{2'}$ with $C_k^{3''}$, and $C_k^{3'}$ with $C_k^{2''}$, and in the F_k^i -handles identify $D_k^{1'}$ with $D_k^{2''}$, $D_k^{2'}$ with $D_k^{1''}$, and $D_k^{3'}$ with $D_k^{3''}$ (see Figure 1, c)). Denote the identification space by X_{k+1} , and let H_{k+1} (resp. F_{k+1}) be the image of $H_k^2 \sqcup H_k^3$ (resp. $F_k^1 \sqcup F_k^2$) under the identification map (see Figure 1, d)). Define the mapping $f_k^{k+1} : X_{k+1} \rightarrow X_k$ by sending a point $(x, i) \in \bar{X}_k^i$ to $x \in X_k$. The obtained map is a 3-to-1 covering projection, and set

$$X_{\text{Sch}} = \varprojlim \{X_k, f_k^{k+1}, \mathbb{N}_0\}.$$

3.3. The Schreier continuum in the Schori example. Consider the Schori solenoid X_{Sch} , and let $x_0 \in H_0 \cap F_0$. For each $k > 0$ there is a unique point $x_k \in X_k$ such that $x_k \in H_k \cap F_k$ and

$$f_0^k(x_k) = f_{k-1}^k \circ f_{k-2}^{k-1} \circ \cdots \circ f_0^1(x_k) = x_0,$$

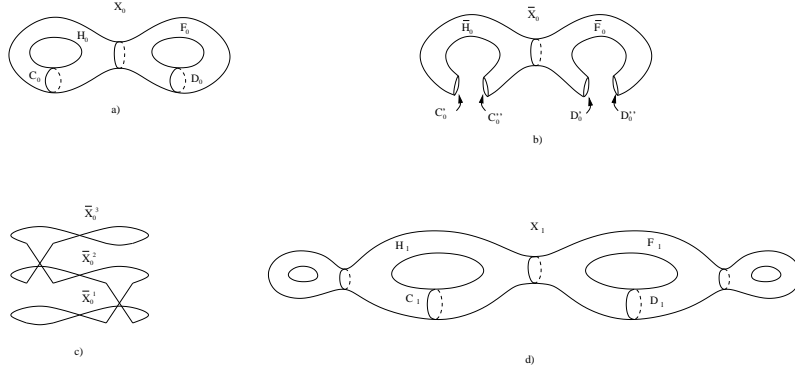


FIGURE 1. Construction of the Schori example: a) Choice of the handles H_0 and F_0 and closed curves C_0 and D_0 in X_0 , b) The cut surface \bar{X}_0 , c) Identifications between \bar{X}_0^i , $i = 1, 2, 3$. Each \bar{X}_0^i is represented by a cut copy of figure 8, and identifications are depicted with straight lines, d) The surface X_1 and the choice of the handles H_1 and F_1 and closed curves C_1 and D_1 .

so $(x_k)_{k=0}^\infty \in X_{\text{Sch}}$. In the sequel we omit the subscript and superscript in the notation for a point in X_{Sch} writing (x_k) instead of $(x_k)_{k=0}^\infty$. Denote by $f_k : X_{\text{Sch}} \rightarrow X_k$ the projection, and by F the fibre $f_0^{-1}(x_0)$ of the bundle $p_0 : X_{\text{Sch}} \rightarrow X_0$ at x_0 .

3.3.1. The group chain and the Schreier continuum. Let $G_0 = \pi_1(X_0, x_0)$, and for $k > 0$ denote $G_k = f_{0*}^k \pi_1(X_k, x_k)$. As it was shown in Section 3, the Schreier continuum for a pointed space $(X_{\text{Sch}}, (x_k))$ is constructed as an inverse sequence of Schreier diagrams Λ_k' of pairs of groups (G_0, G_k) . So the first step in the construction is to compute the group chain $G_0 \supset G_1 \supset G_2 \supset \dots$.

Choose loops a, b, α, β representing the generators of the fundamental group $\pi_1(X_0, x_0)$ in the standard way (see Section 2.3) with a relation

$$\text{rel}_0 = [a, \alpha][b, \beta] = a\alpha a^{-1}\alpha^{-1}b\beta b^{-1}\beta^{-1},$$

where concatenation denotes the usual multiplication of paths. Calculation of G_k by an inductive procedure gives the following. For $k = 0$, distinguish the following subsets of generators of G_0 ,

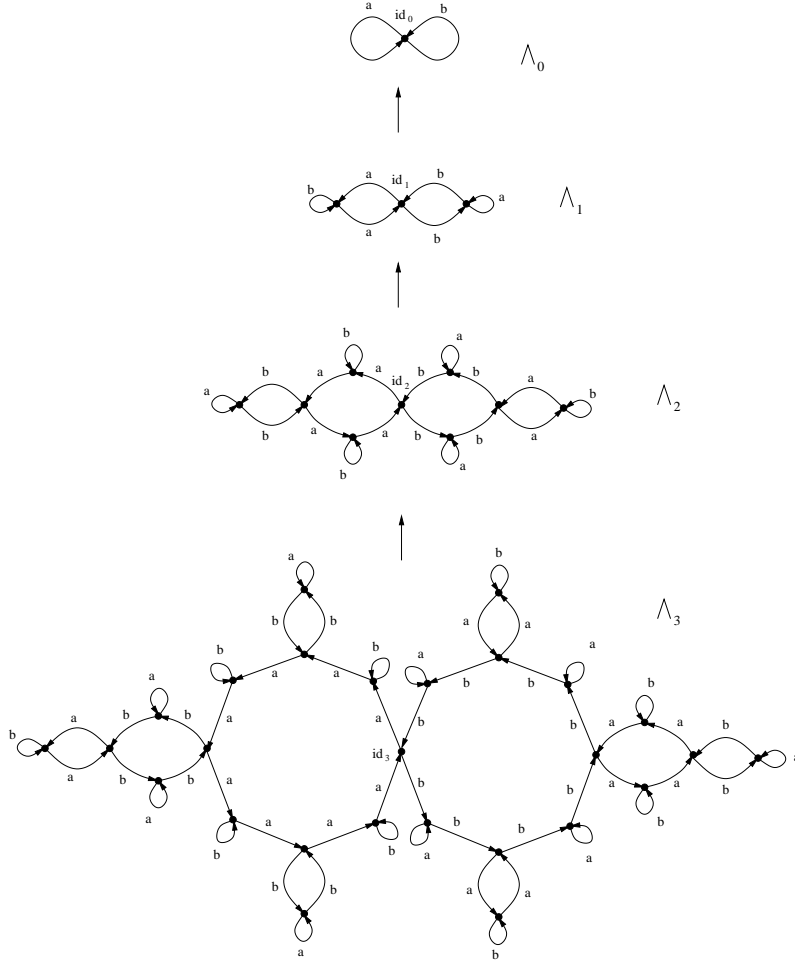
$$\begin{aligned} (3.2a) \quad S_{0ab} &= \emptyset, & S_{0ba} &= \emptyset \\ (3.2b) \quad S_{0a} &= \{a, \alpha\}, & S_{0b} &= \{b, \beta\}. \end{aligned}$$

For $k \geq 1$ define

$$\begin{aligned} S_{kab} &= S_{(k-1)ab} \cup a^{2^{k-1}} S_{(k-1)ab} a^{-2^{k-1}} \cup a^{2^{k-1}} S_{(k-1)ba} a^{-2^{k-1}}, \\ S_{kba} &= S_{(k-1)ba} \cup b^{2^{k-1}} S_{(k-1)ba} b^{-2^{k-1}} \cup b^{2^{k-1}} S_{(k-1)ab} b^{-2^{k-1}}, \\ (3.2c) \quad S_{ka} &= \{a^{2^k}, \alpha\} \cup S_{kab}, \\ S_{kb} &= \{b^{2^k}, \beta\} \cup S_{kba}. \end{aligned}$$

Then for $k \geq 0$ we have

$$G_k = \langle a^{2^k}, \alpha, b^{2^k}, \beta, S_{kab}, S_{kba} \mid \text{rel}_k = \text{id}, \text{rel}_0 = \text{id} \rangle,$$

FIGURE 2. Graphs Λ_i for $i = 0, 1, 2, 3$.

where rel_k is the corresponding relation for the m_k -genus surface. The subgroup G_k has index 3^k in G_0 .

Let \tilde{S}_{0a} and \tilde{S}_{0b} be sets of formal inverses of elements in \tilde{S}_{0a} and \tilde{S}_{0b} respectively, and denote $S = S_{0a} \cup S_{0b} \cup \tilde{S}_{0a} \cup \tilde{S}_{0b}$. For each $k \geq 0$ let Λ'_k be the Schreier diagram of the coset space G_0/G_k constructed with respect to S as in Section 3, and $\Lambda' = \lim\{\Lambda'_k, f_k^{k+1}, \mathbb{N}_0\}$ be the corresponding Schreier continuum. Let $v = (v_k)$ be a vertex of Λ' , and denote by L_v its path-connected component.

3.3.2. The simplified Schreier continuum. We call an edge labelled by λ a λ -edge and write e_λ . Then $s(e_\lambda)$ is the starting vertex of the edge, and $t(e_\lambda)$ is the ending vertex. If $s(e_\lambda) = t(e_\lambda) = v$, e_λ is called a λ -loop based at v . If it is important to keep track the starting point of an edge, we write e_λ^v . A path in Λ' is a continuous map $\gamma : [0, 1] \rightarrow \Lambda'$, so its image is contained in a path-connected component of Λ' . We allow paths to start and end in the interior of edges. A subgraph L' of Λ'

is called path-connected, if for any $v, w \in L'$ there is an edge-path $\gamma \subset L'$ which starts at v and ends at w .

We notice that any vertex $w = (w_k) \in \Lambda'$ is a base point of α -, α^{-1} -, β - and β^{-1} -loops, since by construction any α -, α^{-1} -, β - and β^{-1} -loop based at $w_k \in \Lambda'_k$, $k \in \mathbb{N}_0$ lifts to a loop in Λ'_{k+1} at w_{k+1} labelled by the same letter. Given an exhaustion of a path-connected component L' of w by compact sets, eventually every such loop would be contained in a compact sets and will not make any contribution to the number of ends of L . This motivates the following lemma.

Lemma 3.1. *Let $S' = \{\alpha, \alpha^{-1}, \beta, \beta^{-1}\}$, for $k \in \mathbb{N}_0$ let*

$$\Lambda_k = \Lambda'_k \setminus \bigcup_{w_k \in V(\Lambda'_k)} \left(\bigcup_{\lambda \in S'} \text{int } e_\lambda^{w_k} \right),$$

and $r_k : \Lambda'_k \rightarrow \Lambda_k$ be a retraction which maps each λ -loop $e_\lambda^{w_n}$, where $\lambda \in S'$, to its base point w_n and is the identity on the rest of Λ'_k . Denote by $i_k : \Lambda_k \rightarrow \Lambda'_k$ the inclusion, and define a continuous map $\nu_{k-1}^k : \Lambda_k \rightarrow \Lambda_{k-1}$ by

$$(3.3) \quad \nu_{k-1}^k(x) = r_{k-1} \circ \nu_{k-1}'^k \circ i_k(x).$$

Then $\{\Lambda_k, \nu_{k-1}^k, \mathbb{N}_0\}$ is an inverse sequence of topological spaces, and there is an induced map

$$(3.4) \quad r_\infty : \Lambda' \rightarrow \Lambda = \varprojlim \{\Lambda_k, \nu_k^{k+1}, \mathbb{N}_0\},$$

which is a bijection between the sets of path-connected components of Λ' and Λ . Moreover, for any vertex $v \in \Lambda'$ the path-connected components L'_v and $r(L'_v)$ have the same number of ends.

Proof. The mappings (3.3) satisfy $\nu_i^k = \nu_i^j \circ \nu_j^k$ for any $i < j < k$, and Λ_k are compact, so the inverse limit Λ exists and is non-empty [27]. The retractions r_k satisfy $r_k \circ \nu_k^{k+1} = \nu_k^{k+1} \circ r_{k+1}$ and are surjections onto compact spaces Λ_k , so they induce a continuous surjection (3.4) on the inverse limit spaces. On the other hand, the inclusions i_k induce a continuous mapping $i_\infty : \Lambda \rightarrow \Lambda'$ such that $r_\infty \circ i_\infty = \text{id}_\Lambda$. It follows that (3.4) is a bijection between path-connected components of Λ' and Λ , and $r_\infty(L'_v) = L_v$.

Let d' be a complete length metric on the leaves of Λ' such that the restrictions of ν'_k to leaves is a local isometry. In particular, each edge in Λ' has length 1 in the length structure associated to the metric. Denote by d the induced length metric on the subspace Λ of Λ' . Denote by $B'(w, s)$ and $B(w, s)$ compact balls of radius s about a vertex w with respect to d' and d respectively, and consider the exhaustion $\{B'(v, z + \frac{1}{2})\}_{z \in \mathbb{N}}$ of L'_v by closed balls.

Notice that if $\lambda \in S'$ and e_λ^w is a loop, then for any $x \in \text{int } e_\lambda^w$ we have $d'(w, x) \leq \frac{1}{2}$. Therefore, for any $z \in \mathbb{N}$ the intersection $B'(v, z + \frac{1}{2}) \cap e_\lambda^w$ is either empty or contains the edge e_λ^w . Therefore, there is a bijection between the sets of path-connected components of the complements $L'_v \setminus B'(v, z + \frac{1}{2})$ and $L_v \setminus r_\infty(B'(v, z + \frac{1}{2}))$. The sequence

$$\left\{ r_\infty(B'(v, z + \frac{1}{2})) \right\}_{z \in \mathbb{N}} = \left\{ B(v, z + \frac{1}{2}) \right\}_{z \in \mathbb{N}}$$

is an exhaustion of L_v , and it follows that L'_v and L_v have the same number of ends. \square

In the case of the Schori example Λ_0 is a figure 8, and each Λ_k is the Schreier graph of a coset space F_2/G'_k , where F_2 is a free group on two generators and $G'_k \subset F_2$ is a subgroup of finite index (see Figure 2) with the following set of generators: denote by $\bar{S}_{0a} = \{a\}$, $\bar{S}_{0b} = \{b\}$, for $k > 1$ define S_{kab} and S_{kba} as in (3.2c), and

$$S_{ka} = \{a^{2^k}\} \cup S_{kab}, \quad S_{kb} = \{b^{2^k}\} \cup S_{kba}.$$

Then

$$G'_k = \langle a^{2^k}, b^{2^k}, S_{kab}, S_{kba} \rangle.$$

3.3.3. Ends of special path-connected components in the Schori example. As in Lemma 3.1, we consider the sequence of topological spaces Λ_k , $k \in \mathbb{N}_0$, with length structure induced from Λ'_k and the corresponding length metric d_k . Since Λ_k is compact, d_k is bounded. In fact, we can obtain a more precise estimate, which will be of use later.

For $k \in \mathbb{N}_0$ we denote by id_k the coset $[\text{id}] \in F_2/G'_k$, and by a^m (resp. b^m), $m = 1, \dots, 2^k - 1$, the coset $[a^m]$ (resp. $[b^m]$). Denote by \mathcal{A}_k the connected component of $\Lambda_k - \{\text{id}_k\}$, containing vertices a^1, \dots, a^{2^k-1} , and by \mathcal{B}_k the other connected component. Then $\mathcal{A}_k \cup \{\text{id}_k\}$ and $\mathcal{B}_k \cup \{\text{id}_k\}$ are compact connected subspaces of Λ_k .

Lemma 3.2. *The restriction of the length metric d_k to $\mathcal{A}_k \cup \{\text{id}_k\}$ or to $\mathcal{B}_k \cup \{\text{id}_k\}$ has the least upper bound $N_k = 2^k - \frac{1}{2}$, and for any two positive integers $m, \ell < 2^k$*

$$d_k(a^\ell, a^m) = \min\{|m - \ell|, 2^k - |m - \ell|\},$$

and the same holds for $d_k(b^\ell, b^m)$.

Corollary 3.1. *For any positive integers $m, \ell < 2^{k-1}$*

$$d_k(a^\ell, a^m) = |m - \ell|.$$

Denote by id the point $(\text{id}_k) \in \Lambda$.

Corollary 3.2. *If m, ℓ are positive integers, then for $(a^m), (a^\ell) \in L_{\text{id}}$*

$$d((a^m), (a^\ell)) = |m - \ell|,$$

and there is a corresponding geodesic in L_{id} consisting only of a - or only of a^{-1} -edges.

In the following proposition we will also use a notation a^{-m} for a coset $[a^{-m}] \in F_2/G'_k$, where $m \in \mathbb{N}_0$, so that vertices in F_2/G'_k now have two names since $[a^\ell] = [a^{-m}]$ with possibly $\ell \neq m$. We will see that $(a^{-m}) \neq (a^\ell)$ in L_{id} for any $m, \ell > 0$ in the proof of Proposition 3.1.

Proposition 3.1. *The pathwise connected component L_{id} of Λ has four ends.*

Proof. The idea is to choose a cofinal sequence of compact subsets in L_{id} so that their complements are easy to handle. Such a sequence is a sequence of compact balls $B(\text{id}, N_k)$ where $N_k = 2^k - \frac{1}{2}$ (see Lemma 3.2).

Let $A \subset V(L_{\text{id}})$ be a subset of the set of vertices of L_{id} . We say that a subgraph Γ_A is associated to A , if Γ_A contains A and if Γ_A contains all edges $e_\lambda \in L_{\text{id}}$ with

both starting and ending vertices in A . We claim that for each $k \in \mathbb{N}_0$ and for each of the following subsets of $V(L_{\text{id}})$

$$(3.5) \quad \begin{aligned} A_k^+ &= \{(a^m) \mid m > N_k\}, & A_k^- &= \{(a^{-m}) \mid m > N_k\}, \\ B_k^+ &= \{(b^m) \mid m > N_k\}, & B_k^- &= \{(b^{-m}) \mid m > N_k\} \end{aligned}$$

the associated graph lies in a distinct unbounded connected component of the complement $L_{\text{id}} \setminus B(\text{id}, N_k)$, and each $L_{\text{id}} \setminus B(\text{id}, N_k)$ has at most four unbounded connected components.

Clearly $\Gamma_{A_k^+}$ is path-connected. Since $B(\text{id}, N_k)$ contains an interior point of an edge e_λ if and only if it contains at least one of its vertices,

$$\Gamma_{A_k^+} \cap B(\text{id}, N_k) = \emptyset$$

and so $\Gamma_{A_k^+}$ is contained in a path-connected component of $L_{\text{id}} \setminus B(\text{id}, N_k)$. By Corollary 3.2 geodesics between vertices in A_k^+ are contained in $\Gamma_{A_k^+}$ and they can be of arbitrarily large length, so $\Gamma_{A_k^+}$ is contained in an unbounded path-connected component of the complement. By a similar argument each of the sets A_k^-, B_k^+, B_k^- is contained in an unbounded connected component of the complement $L_{\text{id}} \setminus B(\text{id}, N_k)$.

We show that $\Gamma_{A_k^+}$ and $\Gamma_{A_k^-}$ lie in different path-connected components of $L_{\text{id}} \setminus B(\text{id}, N_k)$.

Assume to the contrary that there is a path γ between (a^m) and (a^{-n}) entirely contained in $L_{\text{id}} \setminus B(\text{id}, N_k)$. Choose $j \in \mathbb{N}_0$ so that for each $i \geq j$ the path γ is contained in the fundamental domain of the covering map $\nu_i : L_{\text{id}} \rightarrow \Lambda_i$, then its projection $\gamma_i = \nu_i \circ \gamma$ is contained in $\Lambda_i \setminus B_i(\text{id}_i, N_k)$, where $B_i(\text{id}_i, N_k)$ is a closed ball about id_i of radius N_k with respect to the length metric d_i on Λ_i . Then

$$\ell(\gamma) \geq \ell_i(\gamma_i) \geq 2^i - (m + n),$$

where $\ell(\gamma)$ and $\ell_i(\gamma_i)$ are the lengths of paths in the length structures associated to d and d_i respectively. Since i can be arbitrary large, $\ell(\gamma)$ cannot be finite. It follows that $\Gamma_{A_k^+}$ and $\Gamma_{A_k^-}$ (and, by a similar argument, $\Gamma_{B_k^+}$ and $\Gamma_{B_k^-}$) are in different path connected components of $L_{\text{id}} \setminus B(\text{id}, N_k)$.

Next, suppose $\Gamma_{A_k^+}$ and $\Gamma_{B_k^+}$ (resp. $\Gamma_{B_k^-}$) are in the same path-connected component of $L_{\text{id}} \setminus B(\text{id}, N_k)$ and choose $j \in \mathbb{N}$ so that a path γ between $(a^m) \in A_k^+$ and $(b^n) \in B_k^+$ (resp. $(b^{-n}) \in B_k^-$) is contained in the fundamental domain of the covering map $\nu_j : L_{\text{id}} \rightarrow \Lambda_j$. Then $\gamma_j = \nu_j \circ \gamma$ is contained in a path-connected component of $\Lambda_j \setminus B_j(\text{id}_j, N_k)$, which implies that either $a^m, b^n \in \mathcal{A}_j$ or $a^m, b^n \in \mathcal{B}_j$ (resp. $a^m, b^{-n} \in \mathcal{A}_j$ or $a^m, b^{-n} \in \mathcal{B}_j$), a contradiction. Therefore, $\Gamma_{A_k^+}$ and $\Gamma_{B_k^+}$ (resp. $\Gamma_{B_k^-}$) are in different path-connected components of $L_{\text{id}} \setminus B(\text{id}, N_k)$. Repeat the same argument for $\Gamma_{A_k^-}$. Conclude that $L_{\text{id}} \setminus B(\text{id}, N_k)$ has at least four unbounded path-connected components.

We show that $L_{\text{id}} \setminus B(\text{id}, N_k)$ has at most 4 path-connected components. Note that each $B(\text{id}, N_k)$ is precisely the fundamental domain of the covering map $\nu_k : L_{\text{id}} \rightarrow \Lambda_k$. Then for $i > k$ the coboundary of the set of vertices $V(B_i(\text{id}_i, N_k))$ consists of exactly 4 edges. Let $v = (v_n) \in L_{\text{id}} \setminus B(\text{id}, N_k)$, and γ be a path in L_{id} between id and v . Choose an integer $j > k$ such that $i > j$ implies that the path γ is contained in the fundamental domain of the covering map $\nu_i : L_{\text{id}} \rightarrow \Lambda_i$, and so projects faithfully on Λ_i . Since $\text{id}_i = \nu_i(\text{id}) \in B_i(\text{id}_i, N_k)$ and $v_i \in \Lambda_i \setminus B_i(\text{id}_i, N_k)$,

the projection $\gamma_i = \nu_i \circ \gamma$ contains an edge of the coboundary of $V(B_i(\text{id}_i, N_k))$. It follows v is in a path-connected component containing one of the sets in (3.5). \square

Let $q_0 = \text{id}_0$, $q_1 = a$ and for $k \in \mathbb{N}_0$, $k > 1$ define

$$(3.6) \quad q_{2k} = b^{2^k} q_{2k-1}, \quad q_{2k+1} = a^{2^k+1} q_{2k}.$$

Then $\nu_{k-1}^k(q_k) = q_{k-1}$, and $q = (q_k)$ is a vertex in Λ .

Lemma 3.3. *A path-connected component L_q of Λ has 1 end.*

Proof. We claim that for each $k > 0$ the set $U_k = L_q \setminus B(q, N_k - \frac{1}{2})$ is path-connected. First notice that for any $i > k$ we have

$$\nu_i \left(B(q, N_k - \frac{1}{2}) \right) = B_i(q_i, N_k - \frac{1}{2}),$$

and if $d(q, v) > N_k - \frac{1}{2}$, then $\nu_i(v) \in B_i(q_i, N_k - \frac{1}{2})$ for at most a finite number of $i \in \mathbb{N}_0$. Denote by \mathcal{Q}_i and \mathcal{P}_i path-connected components of $\Lambda_i \setminus \{\text{id}_i\}$ so that $q_i \in \mathcal{Q}_i$, and notice that if $i > k$ then $\nu_i \left(B(q, N_k - \frac{1}{2}) \right) \subset \mathcal{Q}_i \cup \{\text{id}_i\}$. By construction of Λ_i and the choice of the radius of the balls (see Lemma 3.2) the complement $(\mathcal{Q}_i \cup \{\text{id}_i\}) \setminus \nu_i \left(B(q, N_k - \frac{1}{2}) \right)$ is path-connected.

Let $s = (s_n)$ and $t = (t_n)$ be points in U_k , and γ and δ be paths joining q with s and t respectively. Choose $j > k$ so that

$$\max\{\ell(\gamma), \ell(\delta)\} < N_j - \frac{1}{2},$$

and for any $i > j$ we have $s_i, t_i \in \Lambda_i \setminus B_i(q_i, N_k - \frac{1}{2})$. Then for all $i \geq j$ the paths $\gamma_i = \nu_i \circ \gamma$ and $\delta_i = \nu_i \circ \delta$ are contained in $\mathcal{Q}_i \cup \{\text{id}_i\}$. Choose a path c_j contained in $\mathcal{Q}_j \cup \{\text{id}_j\}$ joining s_j and t_j . Then for any $i > j$ the lift c_i of c_j with an endpoint s_i has t_i as another endpoint. So U_k is path-connected and L_q has one end. \square

Let $q'_0 = \text{id}_0$, $q'_1 = b$ and define for $k \in \mathbb{N}_0$, $k > 1$

$$(3.7) \quad q'_{2k} = a^{2^k} q'_{2k-1}, \quad q'_{2k+1} = b^{2^k+1} q'_{2k},$$

then $\nu_{k-1}^k(q'_k) = q'_{k-1}$, and $q' = (q'_k)$ is a vertex in Λ . By the argument similar to that of Lemma 3.3 the path-connected component $L_{q'}$ has one end.

3.3.4. Other path-connected components in the Schori example. The remaining path-connected components of Λ can be divided into two groups: those containing a so-called ‘dyadic’ point, and those containing a so-called ‘flip-flopping’ point. First we recall and introduce some notation.

As in Lemma 3.3.3, denote by \mathcal{A}_k the path-connected component of $\Lambda_k \setminus \{\text{id}_k\}$ containing cosets $a^i = [a^i] \in G_0/G_k$, and by \mathcal{B}_k the other one. The vertex id_k has three preimages under $\nu_k^{k+1} : \Lambda_{k+1} \rightarrow \Lambda_k$, those being the vertices $\text{id}_{k+1}, a^{2^k}, b^{2^k}$. Then $\mathcal{A}_{k+1} \setminus \{a^{2^k}\}$ consists of three path-connected components: the component \mathcal{A}_{k+1}^+ containing vertices a^i for $i < 2^k$, the component \mathcal{A}_{k+1}^- containing vertices a^i for $i > 2^k$, and the remaining component $\mathcal{T}\mathcal{A}_{k+1}$. Similarly, $\mathcal{B}_{k+1} \setminus \{b^{2^k}\}$ consists of path-connected components \mathcal{B}_{k+1}^+ , \mathcal{B}_{k+1}^- and $\mathcal{T}\mathcal{B}_{k+1}$.

Let $v = (v_k) \in \Lambda$ be a vertex, and assume that $v \notin L_{\text{id}}$. If for some k we have $v_k \in \mathcal{A}_k$ (resp. $v_k \in \mathcal{B}_k$), then either $v_{k+1} \in \mathcal{A}_{k+1}^+ \cup \mathcal{A}_{k+1}^-$ (resp. $v_{k+1} \in \mathcal{B}_{k+1}^+ \cup \mathcal{B}_{k+1}^-$) or $v_{k+1} \in \mathcal{T}\mathcal{B}_{k+1}$ (resp. $v_{k+1} \in \mathcal{T}\mathcal{A}_{k+1}$), and then the following situations are possible:

- (1) $v_k \in \mathcal{A}_k^+ \cup \mathcal{A}_k^- \cup \mathcal{B}_k^+ \cup \mathcal{B}_k^-$ for at most a finite number of k 's, then either $v_k \in L_q$ or $v_k \in L_{q'}$.
- (2) $v_k \in \mathcal{TA}_k \cup \mathcal{TB}_k$ for at most a finite number of indices, then v is called a *dyadic point*. For example, let $j \in \mathbb{N}_0$ be odd, choose $0 \leq \ell_j < 2^j$ and define for $2k > j$

$$\ell_{2k} = \ell_{2k-1} + 2^{2k-1}, \quad \ell_{2k+1} = \ell_{2k}.$$

Then (a^{ℓ_n}) is a dyadic point.

- (3) there is a cofinal subset $I \in \mathbb{N}_0$ such that for any $k \in I'$ we have $v_k \in \mathcal{TA}_k \cup \mathcal{TB}_k$, and the subset $\mathbb{N}_0 \setminus I$ is also cofinal. In this case v is called a *flip-flopping point*.

Lemma 3.4. *Let $v \in \Lambda$ be a dyadic point. Then L_v has 2 ends.*

Proof. Suppose that for $k > n$ we have $v_k \in \mathcal{A}_k$ (resp. $v_k \in \mathcal{B}_k$). Then the distance

$$d_k(v_k, \{a^1, \dots, a^{2^k-1}\}) = \inf\{d_k(v_k, a^\ell) \mid 1 \leq \ell \leq 2^k - 1\}$$

is constant for all $k > n$. Therefore, every dyadic point is in the path-connected component of a point $\tilde{a} = (a^{\ell_k})$, where $0 < \ell_k < 2^k$.

We construct an increasing sequence $K_0 \subset K_1 \subset K_2 \subset \dots$ of compact subsets in L_v such that $\tilde{a} \in K_0$ and for $i > 0$ $L_v \setminus K_i$ has precisely 2 unbounded path-connected components. Set $K_0 = B(\tilde{a}, N_0)$ and suppose K_{i-1} is given. Then there exists $j_i > 0$ such that $K_{i-1} \subset B(\tilde{a}, N_{j_i})$. Construct K_i as follows. Lemma 3.2 together with a simple computation show that $d_k(a_k^\ell, \text{id}_k)$ increases with k , so there exists $n > 0$ such that for all $k \geq n$ the projection $\nu_k(B(\tilde{a}, N_{j_i})) \subset \mathcal{A}_k$. We can also assume that $2^i < 2^{n-2}$. Denote $\pm m = \ell_n \pm (2^{j_i} - 1)$. The coboundary of $V(B_n(a^{\ell_n}, N_{j_i}))$ contains the set of edges $W_n = \{e_a^m, e_{a^{-1}}^m, e_a^{-m}, e_{a^{-1}}^{-m}\} \subset \Lambda_n$, and a possibly empty set $E_n = \{e_{\lambda_1}, \dots, e_{\lambda_{s_n}}\}$. There is a finite set of vertices $V_{E_n} = \{w_1^n, \dots, w_m^n\} \in \Lambda_n$ such that a geodesic δ joining a^{ℓ_n} to w_t contains an edge from E_n . The graph $K' = B_n(a^{\ell_n}, N_{j_i}) \cup \Gamma_{V_{E_n}} \cup E_n$ is pathwise connected. For any $k > n$ there is a unique preimage $\tilde{K}' \subset \Lambda_k$ of K' containing a^{ℓ_k} , and its coboundary is precisely the set $W_k = \{e_a^m, e_{a^{-1}}^m, e_a^{-m}, e_{a^{-1}}^{-m}\} \subset \Lambda_k$. The inverse limit of \tilde{K}' is a subset $K_i \subset L_v$ such that $K_i \ni \tilde{a}$.

Denote by \bar{a}_k^m a vertex in Λ_k corresponding to the coset $[a^m a^{\ell_k}]$, and define

$$(3.8) \quad A_i^+ = \{\bar{a}^m = (\bar{a}_k^m) \mid m > N_{j_i}\}, \quad A_i^- = \{\bar{a}^{-m} = (\bar{a}_k^{-m}) \mid m > N_{j_i}\}.$$

The graphs $\Gamma_{A_k^+}$ and $\Gamma_{A_k^-}$ are clearly path-connected. To see that a path-connected component of $L_v \setminus K_i$ containing such graph is unbounded we show that $\bar{a}^m = \bar{a}^s$ if and only if $m = s$, which implies that $\Gamma_{A_k^\pm}$ contains geodesics of arbitrary length. Indeed, choose $n \in \mathbb{N}_0$ such that $\max\{m, s\} < 2^n$, then for all $k > n$ we have $\bar{a}_k^s = \bar{a}_k^t$ if and only if $s = t$. Next, consider the points $\bar{a}^{2^{j_i}} \in A_i^+$, $\bar{a}^{-2^{j_i}} \in A_i^-$, and assume there exists a path γ contained in $L_v \setminus K_i$ joining $\bar{a}^{2^{j_i}}$ and $\bar{a}^{-2^{j_i}}$. Choose $n > 0$ so big that for $k > n$ the projection $\nu_k(K_i) \subset \mathcal{A}_k$ and $2^{j_i} < 2^{n-2}$. The projection $\gamma_k = \nu_k \circ \gamma$ is a path joining $\bar{a}_k^{2^{j_i}}$ and $\bar{a}_k^{-2^{j_i}}$ and so must contain a sequence of a -edges. Assuming $\bar{a}_k^{2^{j_i}}$ to be a starting point of γ_k , the first edge traversed by γ_k must be e_b , $e_{b^{-1}}$ or e_a . In the two former cases γ contains a loop based at $\bar{a}^{2^{j_i}}$, so we can as well assume that the first edge traversed by γ_k is e_a . By a similar

argument γ_k must contain all a - or all a^{-1} -edges in Λ_k such that $e_\lambda \cap \nu_k(K_i) = \emptyset$. Then we have the following estimate on the length $\ell(\gamma)$ and $\ell_k(\gamma_k)$

$$\ell(\gamma) \geq \ell_k(\gamma_k) \geq 2^k - (2^{j_i} + 1),$$

which implies that γ has unbounded length. Therefore, $\Gamma_{A_k^+}$ and $\Gamma_{A_k^-}$ lie in different path-connected components of $L_v \setminus K_i$. We show that $L_v \setminus K_i$ has precisely two path-connected components. Assume $v \in L_v \setminus K_i$ and let δ be a path joining v with \tilde{a} . Then δ_k contains an edge of W_k , and it follows that v is in a path-connected component of A_i^+ or A_i^- . So there is at most 2 path-connected components. \square

Lemma 3.5. *Let $v \in \Lambda$ be a flip-flopping point. Then L_v has 1 end.*

Proof. We construct a sequence of compact sets $K_0 \subset K_1 \subset K_2 \dots$ such that $K_0 \ni v$ and $L_v \setminus K_i$ has one unbounded path-connected component. Set $K_0 = B(v, N_0)$, and suppose K_{i-1} is given. Then there exists $j_i > 0$ such that $K_{i-1} \subset B(v, N_{j_i})$. Since $d_k(a^{2^k}, \text{id}_k)$ and $d_k(b^{2^k}, \text{id}_k)$ increase with k , there is an $n > 0$ such that for any $k \geq n$ such that $v_k \in \mathcal{TA}_k \cup \mathcal{TB}_k$ we have $B_k(v_k, N_{j_i}) \subset \mathcal{TA}_k \cup \mathcal{TB}_k$. Then $\nu_k(B(v, N_{j_i})) = B_k(v_k, N_{j_i})$. Denote by U_n the path-connected component of $\Lambda_n \setminus B_n(v_n, N_{j_i})$ containing id_n , and let \mathcal{U}_n be a possibly empty set of remaining path-connected components. If $\bar{U}_n \in \mathcal{U}_n$, then $\bar{U}_n \subset \mathcal{TA}_n \cup \mathcal{TB}_n$, and for any $k \geq n$ the preimage $\bar{U}_k \ni v_k$ is a path-connected component of $\Lambda_k \setminus B_k(v_k, N_{j_i})$. Let $W_k = B_k(v_k, N_{j_i}) \cup \mathcal{U}_k$, then for each $k \geq n$ the complement $\Lambda_k \setminus W_k$ is path-connected. Set

$$K_i = \varprojlim \{W_k, \nu_k^{k+1}, k \geq n\}.$$

Let $s = (s_k), t = (t_k) \in L_v \setminus K_i$, and γ and δ be paths joining s and t with v respectively. Let $I \subset \mathbb{N}_0$ be a cofinal subsequence of indices such that if $k \in I$ then $v_k \in \mathcal{TA}_k \cup \mathcal{TB}_k$. Let $M = \max\{\ell(\gamma), \ell(\delta)\}$ and choose $n \in I$ such that $B_n(v_n, M) \subset \mathcal{TA}_n \cup \mathcal{TB}_n$. Then $\gamma_n = \nu_n \circ \gamma$ and $\delta_n = \nu_n \circ \delta$ are contained in $\mathcal{TA}_n \cup \mathcal{TB}_n$, and so either $s_n, t_n \in \mathcal{TA}_n \cup \{a^{2^n}\}$ or $s_n, t_n \in \mathcal{TB}_n \cup \{b^{2^n}\}$. Then there exists a path c_n contained in one of these sets, which joins s_n with t_n . From the construction of Λ_k it follows that for any $k \in I$ such that $k > n$ the lift c_k of c_n such that one of its endpoints is s_k has t_k as another endpoint. So there is a path c in $L_v \setminus K_i$ joining s and t . It follows that L_v has one end. \square

3.3.5. The number of leaves with one or two ends. It was shown in Blanc [1] that if a minimal compact foliated metric space has a residual set of points whose leaves have 2 ends, then any leaf in \mathcal{F} has one or two ends. In the Schori example the leaf L_{id} has four ends, therefore, the minimal foliation of X_{Sch} must have a residual set of points with leaves with one end. That is indeed the case, as Proposition 3.2 shows directly.

Denote by $\mathcal{AB} \subset V(\Lambda)$ the subset of vertices which lie in 4- or 2-ended path-connected components of Λ .

Proposition 3.2. *The set $\mathcal{AB} \subset V(\Lambda)$ of vertices which belong to 4- and 2-ended path-connected components of Λ is meagre.*

Proof. Consider the subset

$$A_0 = \{(a^{\ell_k}) \in V(\Lambda) \mid \ell_{k-1} = \ell_k \pmod{2^{k-1}}\}.$$

All vertices in A_0 lie either in L_{id} or in a 2-ended path-connected component of Λ , the first alternative occurring if (a^{ℓ_k}) is eventually constant. Denote also

$$A_m = \left\{ (v_k) \in V(\Lambda) \mid \min_{(u_k) \in A_0} d((v_k), (u_k)) \leq m \right\},$$

where d denotes metric on the leaves of Λ . We are going to show that for each $m \geq 0$ the set A_m is nowhere dense. For that it is enough to show that A_m is closed, i.e. $\text{Cl}(A_m) = A_m$, since A_m has empty interior by minimality of 1-ended path-connected components.

Lemma 3.6. *The subset $A_m \subset V(\Lambda)$ is closed.*

Proof. First let $m = 0$. Let $\{(a^{\ell_k})_s\}$ be a sequence of elements in A_0 converging to a point $(\bar{a}_k) \in \text{Cl}(A_0)$. Then for each $k \in \mathbb{N}$ there exists N_k such that for all $s > N_k$ and all $i < k$ we have

$$a_s^{\ell_i} = \bar{a}_i.$$

It follows that each $\bar{a}_k \in \nu_k(A_0)$ and so $(\bar{a}_k) \in A_0$.

Let $m > 0$, and $\{(v_k)_s\} \in A_m$ be a sequence converging to $(\bar{v}_k) \in \text{Cl}(A_m)$. Then

$$\min_{(u_k) \in A_0} d((\bar{v}_k), (u_k)) \leq m$$

since otherwise there exists an open ball around \bar{v}_k , for example, of radius $\frac{1}{2}$, which does not contain points of $\{(v_k)_s\}$. Thus $(\bar{v}_k) \in A_m$. \square

Similarly, denote

$$B_0 = \{(b^{\ell_k}) \in V(\Lambda) \mid b^{\ell_{k-1}} = b^{\ell_k} \pmod{2^{k-1}}\}$$

and for $m > 0$ set

$$B_m = \left\{ (v_k) \in V(\Lambda) \mid \min_{(u_k) \in B_0} d((v_k), (u_k)) \leq m \right\}.$$

Then for $m \geq 0$ the set B_m is nowhere dense, and $A_m \cup B_m$ is also nowhere dense. Any 2-ended path-connected component intersects $A_0 \cup B_0$, so

$$\mathcal{AB} = \bigcup_{m=0}^{\infty} (A_m \cup B_m),$$

and \mathcal{AB} is a meagre set. \square

Proposition 3.2 implies that the set of vertices in Λ which lie in 1-ended path-connected components is residual, so the generic leaf in X_{Sch} has 1 end [9].

Remark 3.1. The number of path-connected components of Λ with 2 ends is uncountable. Consider the dyadic solenoid

$$\Sigma = \varprojlim \{\mathbb{S}^1, f_{k-1}^k, \mathbb{N}_0\},$$

where $f_{k-1}^k : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is a 2-fold covering. The foliation of Σ by path-connected components has an uncountable infinity of leaves. Recall from Section 3.3.3 that there is a subset of leaves with 2 ends such that each leaf contains a dyadic point (a^{ℓ_k}) , and notice that to $(a^{\ell_k}) \in X_{\text{Sch}}$ one can associate a unique point in Σ . Using an argument on the existence of paths as in Section 3.3.3 one sees that if two points in Σ represent different path-connected components, then the corresponding points $(a^{\ell_i}) \in X_{\text{Sch}}$ represent different path-connected components of X_{Sch} . Therefore, X_{Sch} has an uncountable infinity of leaves with 2 ends.

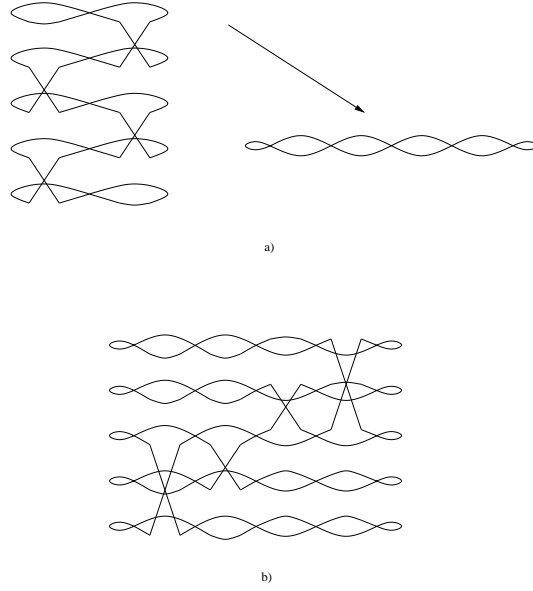


FIGURE 3. Schreier diagrams for the weak solenoid with an 8-ended path component: a) the 5-fold covering space Λ_1 of the figure 8, b) the 5-fold covering space Λ_2 of Λ_1

3.4. Generalised Schori example: solenoid with a $2n$ -ended path-connected component. The Schori example can be generalised to obtain a solenoid with a $2n$ -ended path component, for any $n \in \mathbb{N}$. The construction is based on that of the original Schori example.

3.4.1. The generalised Schori example with a $4n$ -ended path-connected component.

Let X_0 be a genus 2 surface with two n -handles H_1 and H_2 distinguished, let C_1 and C_2 be closed meridional curves in the interior of H_1 and H_2 respectively. The handles H_1 and H_2 intersect along their boundary curves. Cut the handles along C_1 and C_2 , pull them apart by an appropriate homeomorphisms to obtain the cut surface \hat{X}_0 . Set $\bar{X}_0 = \text{Cl}(\hat{X}_0)$ and consider copies $\bar{X}_0^{(i)}$, $i = 1, \dots, 2n+1$ of \bar{X}_0 with handles $\bar{H}_1^{(i)}$ and $\bar{H}_2^{(i)}$ and pairs of boundary circles $C_1^{(i)'}$ and $C_1^{(i)''}$, and $C_2^{(i)'}$ and $C_2^{(i)''}$ in each handle. Make identifications so that for $i = 1, \dots, n$ $C_1^{(2i-1)'}$ is identified with $C_1^{(2i)''}$, $C_1^{(2i-1)''}$ is with $C_1^{(2i)'}$, $C_2^{(2i)'}$ with $C_2^{(2i+1)''}$, $C_2^{(2i)''}$ with $C_2^{(2i+1)'}$ (see Figure 3 for $n = 2$), and other cuts are identified trivially, i.e. $C_t^{(s)'} is identified with $C_t^{(s)''}$. Denote the obtained genus $2n+2$ surface by X_1 . Distinguish $2n$ handles H_1, \dots, H_{2n} (those produced as a result of non-trivial identifications). Define $f_0^1 : X_1 \rightarrow X_0$ as in the Schori example.$

Let X_k be a genus $((2n+1)^k - 2kn)$ -surface, let $2n \cdot 2^k$ -handles H_1, \dots, H_{2n} in X_k be distinguished, and let C_1, \dots, C_{2n} be simple closed curves in the handles. Following the procedure described in the first step obtain $\bar{X}_k^{(i)}$, $i = 1, \dots, 2n+1$ cut surfaces, so that in each of them the cut handles $\bar{H}_j^{(i)}$, $j = 1, \dots, 2n$, are

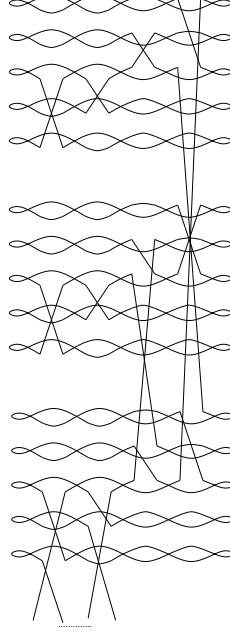


FIGURE 4. Schreier diagrams for the weak solenoid with an 8-ended path component: the 5-fold covering space Λ_3 of Λ_2 , three first layers.

distinguished, and $C_j^{(i)'}$ and $C_j^{(i)''}$ are boundary curves contained in $\bar{H}_j^{(i)}$. Then make identifications as follows (see Figure 4 for $n = 2$):

- (1) for $1 \leq s \leq n$ identify $C_s^{(s)'} with $C_s^{(n+1)''}$, and $C_s^{(s)''}$ with $C_s^{(n+1)'}$,$
- (2) for $n+2 \leq s \leq n+1$ identify $C_{s-1}^{(s)'} with $C_{s-1}^{(n+1)''}$, and $C_{s-1}^{(s)''}$ with $C_{s-1}^{(n+1)'}$,$
- (3) identify other handles trivially.

Denote the resulting identification space X_{k+1} , and define $f_k^{k+1} : X_{k+1} \rightarrow X_k$ as in the Schori example. Obtain X_{Sch}^{4n} as the inverse limit of the sequence of covering spaces f_{k-1}^k .

Choose a point $x_0 \in H_1 \cap H_2$. For each $k > 0$ there is a unique point x_k such that $f_0^k(x_k) = f_{k-1}^k \circ \dots \circ f_0^1(x_k) = x_0$ and $x_0 \in H_n \cap H_{n+1}$, where H_i are handles in X_k . Denote $x = (x_k)$. Using the method of the Schreier continuum one obtains that the path-connected component L_x has $4n$ ends, topologically almost all leaves in X_{Sch}^{4n} have 1 end and there is an uncountable infinity of path-connected components with 2 ends.

3.4.2. The generalised Schori example with a $4n + 2$ -ended path component. Such an example is obtained by a slight alteration of the one in Section 3.4.1: at the first step one identifies $2n$ cut copies of a genus 2 surface X_0 along $2n - 1$ cuts to obtain a genus $2n$ surface X_1 . On the k -th step one considers a genus $((2n)^k - k(2n - 1))$ -surface X_k with $(2n - 1)$ handles, which are identified as in Section 3.4.1, only there is one less identification of the second type. The resulting space X_{k+1} is a covering space of X_0 .

So one obtains a weak solenoid X_{Sch}^{4n+2} with one $4n+2$ -ended path component, which also has path-connected components with 1 and 2 ends. By the same argument as in Section 3.4.1 the generic path-connected component has 1 end, and there is an uncountable infinity of path-connected components with 2 ends.

3.5. Rogers-Tollefson example revisited. The example introduced by Rogers and Tollefson [28] (see also Example 3.2) is that of a non-homogeneous solenoid

$$K_\infty = \varprojlim \{K_i, f_{i-1}^i, \mathbb{N}_0\},$$

where K_i is the Klein bottle and f_{i-1}^i is a 2-fold covering map constructed as follows (see Fokkink and Oversteegen [14]).

Represent the 2-torus as the quotient space $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ and define

$$(3.9) \quad \bar{f} : \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} : (x, y) \mapsto (x, 2y),$$

a 2-fold covering map of the torus by itself. The Klein bottle is the quotient of $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ under the map $i : (x, y) \mapsto (x + \frac{1}{2}, -y)$, and the covering projection (3.9) induces a 2-fold covering map $f : K \rightarrow K$. Define the curves b and a in K by $b(t) = (\frac{1}{2}t, 0)$ and $a(t) = (0, t)$ for $t \in [0, 1]$. Then the homotopy classes $a = [a]$ and $b = [b]$ generate the fundamental group $\pi_1(K, \{0\})$ with a single relation $bab^{-1} = a^{-1}$. The induced map of the fundamental group is

$$f_\# : \pi_1(K, \{0\}) \rightarrow \pi_1(K, \{0\}) : (b, a) \mapsto (b, a^2),$$

with the corresponding decreasing chain of groups $G_k = \langle b, a^{2^k} \rangle$. Although f is a regular covering map, the composition of f with itself is not regular. The kernel of the group chain $\{G_k\}$ depends on the point, more precisely, for $\text{id} = (\{0\})$ we have $K_{\text{id}} = \langle b \rangle$ and for any other point $x \in f_0^{-1}(\{0\})$ which does not belong to the path-connected component of id we have $K_x = \langle b^2 \rangle$. The Schreier diagrams for the pairs (G_0, G_k) with $k = 0, 1, 2, 3$ are shown in Figure 5.

Using the argument on the lengths of paths similar to the ones in Section 3.3 we obtain that the path-connected component of the point $\text{id} \in K_\infty$ has 1 end, and any other path-connected component in K_∞ has 2 ends.

3.6. An example with non self-similar G_0 -action. We modify the Schori example so as to obtain a weak solenoid with non self-similar action of the fundamental group G_0 on the fibre. We first explain how self-similar actions arise in our context.

Recall the definition of a self-similar group action [26]. Let S be a finite alphabet, and S^* be the set of all finite words over S , including the empty word. The set S^* can be thought of as a vertex set of a rooted tree (where the root is given by the empty word), where vertices v and v' are joined by an edge if and only if $v' = vs$ for some $s \in S$. If $\ell(v)$ is the length of the word $v \in S^*$, then v and v' are joined by an edge if and only if their lengths differ by 1. Let G_0 be a group acting faithfully on X^* . Then the action of G_0 is *self-similar* if and only if for every $s \in S$ and $g \in G$ there exist $t \in S$ and $h \in G$ (necessarily unique) such that

$$g(sv) = th(v) \text{ for all } v \in S^*.$$

A nice property of the tree X^* which reflects its self-similarity, is that there is a projection $xv \mapsto v$, which maps a subtree starting at vertex of length 1 to the whole tree.

Now suppose a weak solenoid $X_\infty = \varprojlim \{f_{n-1}^n : X_n \rightarrow X_{n-1}, \mathbb{N}\}$ is given by a sequence of covering maps of constant degree q , and let $x_0 \in X_0$. Choose an

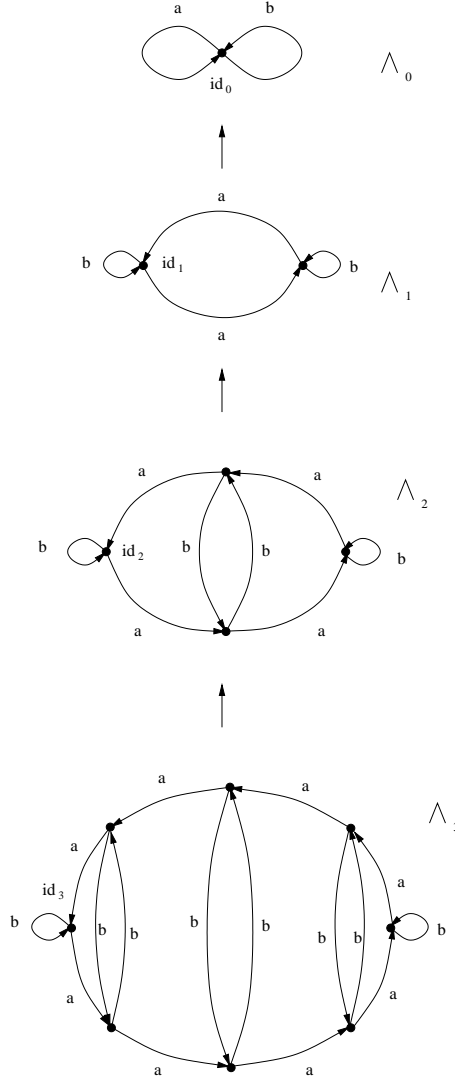


FIGURE 5. Schreier diagrams for the Rogers-Tollefson solenoid for $i = 0, 1, 2, 3$

alphabet S on q symbols. Then we can code preimages of x_0 in X_n by words in S^* of length n , thus associating to the solenoid a rooted tree. Then the fundamental group G_0 acts on S^* in an obvious way, and one can come up with many examples where this action is self-similar. Using the associated tree, one can compute end structures of leaves by methods in [3], using the relation of self-similar actions to finite automata. We now present an example which does not fit into the setting of self-similar actions, but where the number of ends of leaves can be computed by our method.

The most obvious way to construct such an example is to construct a weak solenoid as the inverse limit space of covering maps of variable degree, such that the

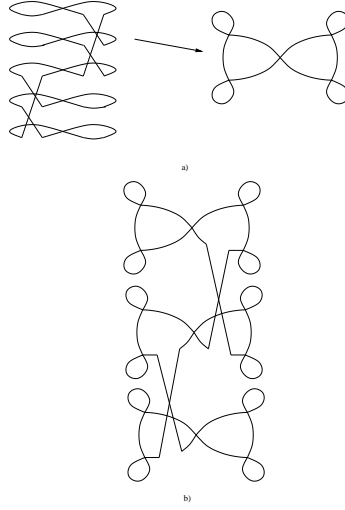


FIGURE 6. Schreier diagrams for the non-selfsimilar solenoid for $i = 0, 1, 2, 3$

situation cannot be reduced to the setting of a self-similar action of the fundamental group on the associated subtree by choosing a subsequence or getting rid of a finite number of factors in the sequence. For that we will alternate 5-to-1 and 3-to-1 coverings of a genus 2 surface as follows. Consider a sequence

$$5 \ 3 \ 3 \ 5 \ 3 \ 3 \ 3 \ 5 \ 3 \ 3 \ 3 \ 3 \ 5 \ 3 \dots$$

where the number 5 occurs infinitely many times, and the number of occurrences of 3 between two occurrences of 5 grows by 1 at each step. Each number in the sequence would correspond to the degree of a covering map $f_{i-1}^i : X_i \rightarrow X_{i-1}$, that is, $f_0^1 : X_1 \rightarrow X_0$ is a 5-fold cover, $f_1^2 : X_2 \rightarrow X_1$ is a 3-fold cover and so on (see Fig. 6). The action of the fundamental group G_0 on the fibre of the corresponding solenoid $X_\infty = \lim_{\leftarrow} \{f_{i-1}^i : X_i \rightarrow X_{i-1}\}$ is non-selfsimilar. The sequence of Schreier diagrams for this solenoid is presented in Fig. 7.

We use the method of the Schreier continuum to compute the number of ends of leaves in this solenoid.

Proposition 3.3. *Let X_∞ be a weak solenoid as above. Then leaves of X_∞ have the following end structures.*

- (1) *There is a single leaf with 4 ends.*
- (2) *There is a residual set of leaves with 1 end.*
- (3) *There is an uncountable meager set of leaves with 2 ends.*

Proof. By a similar argument to the one in Proposition 3.1, the path-connected component containing the point (id_n) , where id_n denotes the coset of G/G_n containing the identity, has 4 ends. Let $q_0 = \text{id}_0$, $q_1 = a$. For $k \in \mathbb{N}_0$ q_{k-1} has a finite number of preimages under f_{k-1}^k . Let q_k be the one at the maximal distance from id_k . Then by argument similar to the one in Lemma 3.3 one shows that the path-connected component containing (q_k) has 1 end. Similarly, we can choose (v_k) in such a way that $v_k = a^{\ell_k}$, and the distance between v_k and id_k eventually grows. Then by an argument similar to that in Lemma 3.4 the path-connected component

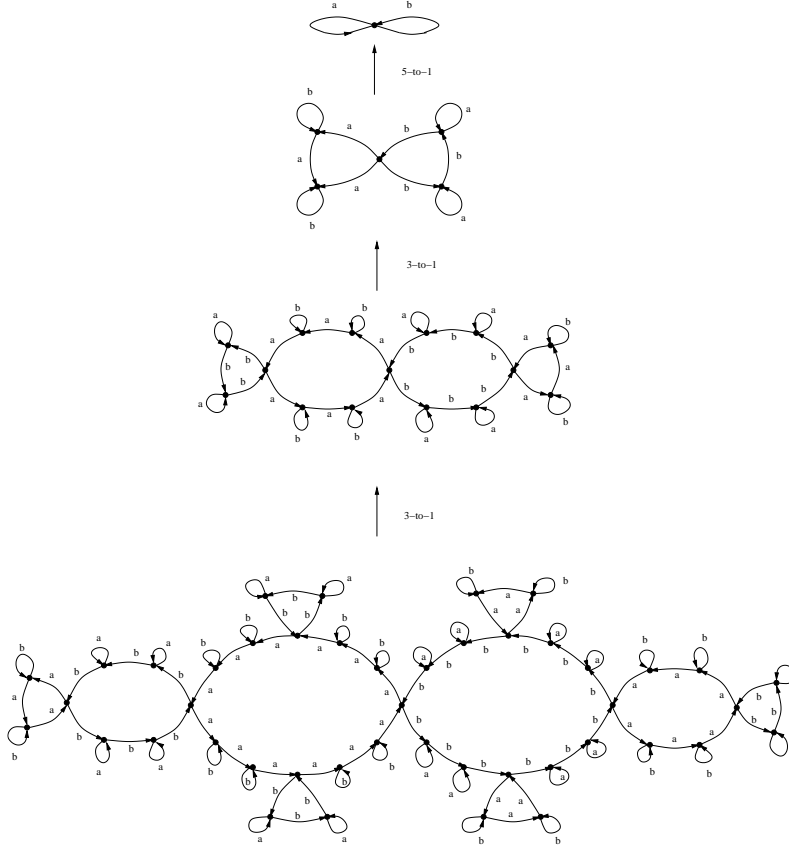


FIGURE 7. Schreier diagrams for the non-selfsimilar solenoid for $i = 0, 1, 2, 3$

of (v_k) has 2 ends. As before, constructing a map from the set of leaves of a solenoid over a circle given by a suitable sequence of integers, one can show that the number of 2-ended path-connected components in X_∞ is uncountable.

We now have to show that every path-connected components in X_∞ has 1, 2 or 4 ends. Then it follows by the result of Blanc [1] that generically leaves in X_∞ have 1 end. We use an argument similar to that in Section 3.3.4. Basically, we need to compute the points whose projections are at variable lengths from id_k , $k \in \mathbb{N}_0$.

Let ν_k^{k+1} be a 3-fold cover. As in Section 3.3.3, denote by \mathcal{A}_k the path-connected component of $\Lambda_k \setminus \{\text{id}_k\}$ containing cosets $a^i = [a^i] \in G/G_k$, and by \mathcal{B}_k the other one. The vertex id_k has three preimages under $\nu_k^{k+1} : \Lambda_{k+1} \rightarrow \Lambda_k$, those being the vertices $\text{id}_{k+1}, a^{2^k}, b^{2^k}$. Then $\mathcal{A}_{k+1} \setminus \{a^{2^k}\}$ consists of three path-connected components: the component \mathcal{A}_{k+1}^+ containing vertices a^i for $i < 2^k$, the component \mathcal{A}_{k+1}^- containing vertices a^i for $i > 2^k$, and the remaining component \mathcal{TA}_{k+1} . Similarly, $\mathcal{B}_{k+1} \setminus \{b^{2^k}\}$ consists of path-connected components \mathcal{B}_{k+1}^+ , \mathcal{B}_{k+1}^- and \mathcal{TB}_{k+1} .

Let ν_k^{k+1} be a 5-fold cover. As before, denote by \mathcal{A}_k the path-connected component of $\Lambda_k \setminus \{\text{id}_k\}$ containing cosets $a^i = [a^i] \in G/G_k$, and by \mathcal{B}_k the other one. The vertex id_k has five preimages under $\nu_k^{k+1} : \Lambda_{k+1} \rightarrow \Lambda_k$, those being the

vertices id_{k+1} and cosets a^m, a^{2m}, b^m, b^{2m} . Then $\mathcal{A}_{k+1} \setminus \{a^m, a^{2m}\}$ consists of five path-connected components: the components \mathcal{A}_{k+1}^j , $j = 1, 2, 3$, such that \mathcal{A}_{k+1}^j contains vertices a^i for $(j-1)m < i < jm$, and the components \mathcal{TA}_{k+1}^s , $s = 1, 2$, such that a^{sm} is in the closure of \mathcal{TA}_{k+1}^s . Similarly, $\mathcal{B}_{k+1} \setminus \{b^m, b^{2m}\}$ consists of path-connected components \mathcal{B}_{k+1}^j , $j = 1, 2, 3$, and \mathcal{TB}_{k+1}^s , $s = 1, 2$.

Let $v = (v_k) \in \Lambda$ be a vertex, and assume that $v \notin L_{\text{id}}$. If for some k we have $v_k \in \mathcal{A}_k$ (resp. $v_k \in \mathcal{B}_k$), then if ν_k^{k+1} is 3-fold, either $v_{k+1} \in \mathcal{A}_{k+1}^+ \cup \mathcal{A}_{k+1}^-$ (resp. $v_{k+1} \in \mathcal{B}_{k+1}^+ \cup \mathcal{B}_{k+1}^-$) or $v_{k+1} \in \mathcal{TB}_{k+1}$ (resp. $v_{k+1} \in \mathcal{TA}_{k+1}$). If ν_k^{k+1} is 5-fold, either $v_{k+1} \in \mathcal{A}_{k+1}^j$, $j = 1, 2, 3$ (resp. $v_{k+1} \in \mathcal{B}_{k+1}^j$, $j = 1, 2, 3$), or $v_{k+1} \in \mathcal{TB}_{k+1}^s$, $s = 1, 2$, (resp. $v_{k+1} \in \mathcal{TA}_{k+1}^s$, $s = 1, 2$).

Then the following situations are possible:

- (1) $v_k \in \mathcal{A}_k^+ \cup \mathcal{A}_k^- \cup \mathcal{B}_k^+ \cup \mathcal{B}_k^-$ if ν_{k-1}^k is 3-fold and otherwise $v_k \in \mathcal{A}_k^1 \cup \mathcal{A}_k^2 \cup \mathcal{A}_k^3 \cup \mathcal{B}_k^1 \cup \mathcal{B}_k^2 \cup \mathcal{B}_k^3$ for at most a finite number of k 's, then argument similar to that in Lemma 3.3 shows that L_v has 1 end.
- (2) $v_k \in \mathcal{TA}_k \cup \mathcal{TB}_k$, if ν_{k-1}^k is 3-fold, and otherwise $v_k \in \mathcal{TA}_k^1 \cup \mathcal{TA}_k^2 \cup \mathcal{TB}_k^1 \cup \mathcal{TB}_k^2$ for at most a finite number of indices, then v is called a *dyadic* point, and by an argument similar to that in Lemma 3.4 L_v has 2 ends.
- (3) there is a cofinal subset $I \in \mathbb{N}_0$ such that for any $k \in I'$ we have $v_k \in \mathcal{TA}_k \cup \mathcal{TB}_k$ if ν_{k-1}^k is 3-fold, and otherwise $v_k \in \mathcal{TA}_k^1 \cup \mathcal{TA}_k^2 \cup \mathcal{TB}_k^1 \cup \mathcal{TB}_k^2$, and the subset $\mathbb{N}_0 \setminus I$ is also cofinal. In this case v is called a *flip-flopping* point. Then by an argument similar to that in Lemma 3.5 L_v has 1 end.

□

4. CONCLUSIONS

For a foliated bundle $p : E \rightarrow B$ with foliation \mathcal{F} we have introduced the notion of the *Schreier continuum* Λ which can be thought of as the union of holonomy graphs of leaves of the minimal set of \mathcal{F} with suitable topology, and have shown that in the case when the minimal set of \mathcal{F} is transversely a Cantor set, a great deal can be said about asymptotic properties of leaves in the minimal set by means of the study of the associated inverse limit representation of Λ .

In particular, using the method of the Schreier continuum we have shown that, given $n > 1$, based on the construction of Schori [30], one can construct a solenoid where a single leaf has $2n$ ends, there is an uncountable infinity of leaves with 2 ends which form a meager subset, and an uncountable infinity of leaves with 1 end, which form a residual subset. We have also shown that the method of Schreier continuum can be used to analyse quite complicated examples, for example, solenoids with non self-similar action of the fundamental group on the fibre.

Blanc [1] proves that if a residual set of points in a foliated space E has leaves with 2 ends, then almost every leaf in E (with respect to a harmonic measure) has 2 ends. Blanc also announces an example showing that this need not be the case if E has a residual set of points with leaves with 1 end. It would be of interest to determine whether this can occur in solenoids.

Question 1. *For the structure of ends in a solenoid, does topologically “almost all” imply measure-theoretically “almost all” ?*

REFERENCES

- [1] E. Blanc, Laminations minimales résiduellement à 2 bouts. *Comment. Math. Helv.* **78** (2003), no. 4, 845–864.
- [2] E. Blanc, Propriétés génériques des laminations. PhD thesis, Université de Claude Bernard-Lyon 1, Lyon 2001.
- [3] I. Bondarenko, D. D'Angeli, and T. Nagnibeda, Ends of Schreier graphs of self-similar groups, preprint, [http : //ibondarenko.110mb.com/research/articles/EndsSchreier.pdf](http://ibondarenko.110mb.com/research/articles/EndsSchreier.pdf).
- [4] B. H. Bowditch, *A course on geometric group theory*. Mathematical Society of Japan, Tokyo 2006.
- [5] R. Bowen, and J. Franks, The periodic points of maps of the disk and the interval. *Topology*, **15** (1976), no. 4, 337–342.
- [6] C. Camacho, and A. Lins Neto, *Geometric theory of foliations*. Birkhäuser Boston Inc., Boston, MA 1985.
- [7] A. Candel, and L. Conlon, *Foliations*. I. Grad. Stud. in Math., 23. AMS, Providence, RI 2000.
- [8] Candel, A.; Conlon, L., *Foliations*. II. Grad. Stud. in Math., 23. AMS, Providence, RI 2000.
- [9] J. Cantwell, and L. Conlon, Generic leaves. *Comment. Math. Helv.* **73** (1998), no. 2, 306–336.
- [10] A. Clark, Linear flows on κ -solenoids. *Topology Appl.* **94** (1999), no. 1-3, 27–49.
- [11] A. Clark, and S. Hurder, Embedding matchbox manifolds, *Topology Appl.* **158** (2011), no. 11, 1249–1270.
- [12] A. Clark, and S. Hurder, Homogeneous matchbox manifolds, to appear in *Trans. Amer. Math. Soc.*
- [13] D. D'Angeli, A. Donno, M. Matter, and T. Nagnibeda, Schreier graphs of the Basilica group, *J. Mod. Dyn.*, **4** (2010), 167–205.
- [14] R. Fokkink, and L. Oversteegen, Homogeneous weak solenoids. *Trans. Amer. Math. Soc.* **354** (2002), no. 9, 3743–3755.
- [15] H. Freudenthal, Über die Enden topologischer Räume und Gruppen. *Math. Zeit.* **33** (1931), 692–713.
- [16] L. Garnett, Foliations, the ergodic theorem and Brownian motion, *J. Funct. Anal.* **51** (1983), no. 3, 285–311.
- [17] É. Ghys, Topologie des feuilles génériques [Topology of generic leaves]. *Ann. of Math.* **141** (1995), no. 2, 387–422.
- [18] J.-M. Gambaudo, and C. Tresser, Diffeomorphisms with infinitely many strange attractors. *J. Complexity*, **6** (1990), no. 4, 409–416.
- [19] J.-M. Gambaudo, D. Sullivan, and C. Tresser, *Infinite cascades of braids and smooth dynamical systems*, *Topology*, **33** (1994), no. 1, 85–94.
- [20] A. Hatcher, *Algebraic topology*. Cambridge University Press, Cambridge 2002.
- [21] H. Hopf, Enden offener Rume und unendliche diskontinuierliche Gruppen. *Comment. Math. Helv.* **16** (1944), 81–100.
- [22] I. Kan, Strange attractors of uniform flows. *Trans. Amer. Math. Soc.*, **293** (1986), no. 1, 135–159.
- [23] L. Markus, and K. R. Meyer, Periodic orbits and solenoids in generic Hamiltonian dynamical systems. *Amer. J. Math.*, **102** (1980), no. 1, 25–92.
- [24] W. S. Massey, A basic course in algebraic topology, GTM 127, Springer-Verlag, New York, 1991.
- [25] M. C. McCord, Inverse limit sequences with covering maps. *Trans. Amer. Math. Soc.* **114** (1965), 197–209.
- [26] V. Nekrashevych, *Self-similar groups*, Mathematical Surveys and Monographs, **117**, AMS, Providence RI 2005.
- [27] L. Ribes, and P. Zalesskii, *Profinite groups*, Springer-Verlag, Berlin 2000.
- [28] J. T. Rogers, Jr., and J. L. Tollefson, Involutions on solenoidal spaces. *Fund. Math.* **73** (1971/72), no. 1, 11–19.
- [29] R. Sacksteder, On the existence of exceptional leaves in foliations of co-dimension one, *Ann. Inst. Fourier (Grenoble)* **14** (1964), 221–225.
- [30] R. Schori, Inverse limits and homogeneity, *Trans. Amer. Math. Soc.* **124** (1966), 533–539.
- [31] P. Scott, Ends of pairs of groups, *J. Pure Appl. Algebra* **11** (1977/78), no. 1-3, 179–198.
- [32] P. Scott, Subgroups of surface groups are almost geometric, *J. London Math. Soc.* **17** (1978), 555–565.

- [33] E. S. Thomas, Jr., One-dimensional minimal sets. *Topology*, **12** (1973), 233–242.
- [34] S. Willard, *General topology*. Dover Publications, Inc., Mineola, New York 2004.

Received September 28, 2011

Revised version received January 6, 2012

ALEX CLARK, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF LEICESTER, UNIVERSITY ROAD,
LEICESTER LE1 7RH, UNITED KINGDOM
E-mail address: `adc20@le.ac.uk`

ROBBERT FOKKINK, DIAM PROBABILITY, TU DELFT, MEKELWEG 4, 2628CD DELFT, NETHER-
LANDS
E-mail address: `R.J.Fokkink@tudelft.nl`

OLGA LUKINA, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF LEICESTER, UNIVERSITY ROAD,
LEICESTER LE1 7RH, UNITED KINGDOM
E-mail address: `ollukina940@gmail.com`